302(7): Derivation of the Classical and Relativistic Binet Equations of Orbits.

Consider the Leibniz equation:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2}. \quad -(1)$$

Use the change of variable:

$$u = \frac{1}{r}. \quad -(2)$$

It follows that:

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = \frac{du}{dr}\frac{dr}{d\phi} \quad -(3)$$

$$= -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\phi} \quad -(4)$$

By conservation of angular momentum:

$$\frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(5)$$

so

$$\frac{du}{d\phi} = -\frac{m}{L}\dot{r}, \quad -(6)$$

and

$$\frac{d^2u}{d\phi^2} = \frac{d}{d\phi}\left(-\frac{m}{L}\dot{r}\right) = \frac{dt}{d\phi}\frac{d}{dt}\left(-\frac{m}{L}\dot{r}\right) \quad -(7)$$

$$= -\frac{m^2}{L^2}r^2\ddot{r}$$

So

$$\ddot{r} = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\phi^2} \quad -(8)$$

and

$$r\dot{\phi}^2 = \frac{L^2}{m^2}u^3 \quad -(9)$$

From eqs. (1), (8) and (9):

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2} F(r) \quad -(10)$$

here

$$F(r) = -\frac{\partial U}{\partial r} = -\frac{mG}{r^2} \quad -(11)$$

so

$$\boxed{\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}} \quad -(12)$$

This is the classical Binet equation for the force (11). It is valid for all $F(r)$ and all orbits. Here

$$\alpha = \frac{L^2}{m^2 mG} \quad -(13)$$

, the half right latitude. By inspection, it is seen that the conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(14)$$

the solution of eq. (12).

As shown in note 402(5), the relativistic Leibniz equation is:

$$\gamma(\ddot{r} - r\dot{\phi}^2) + \dot{r}\frac{d\gamma}{dt} = -\frac{mG}{r^2} \quad -(15)$$

$$= F(r) = -\frac{\partial U(r)}{\partial r}$$

Now note that:

$$\frac{d}{dt}(\gamma\dot{r}) = \gamma\ddot{r} + \dot{r}\frac{d\gamma}{dt} \quad -(16)$$

So eq. (15) is:

$$\frac{d}{dt}(\gamma \dot{r}) - \gamma r \dot{\phi}^2 = -\frac{mG}{r^2} \quad -(17)$$

The relativistic angular momentum is conserved:

$$\frac{dL}{dt} = 0 \quad -(18)$$

here

$$L = \gamma m r^2 \dot{\phi} \quad -(19)$$

so

$$\frac{d\phi}{dt} = \frac{L}{\gamma m r^2} \quad -(20)$$

and

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\phi} = -\frac{\gamma m}{L}\dot{r} \quad -(21)$$

it follows that:

$$\frac{d^2 u}{d\phi^2} = \frac{d}{d\phi}\left(-\frac{\gamma m}{L}\dot{r}\right) = -\frac{m}{L}\frac{dt}{d\phi}\frac{d}{dt}(\gamma \dot{r})$$

$$= -\frac{\gamma m^2 r^2}{L^2}\frac{d}{dt}(\gamma \dot{r}) \quad -(22)$$

So

$$\frac{d}{dt}(\gamma \dot{r}) = -\frac{L^2}{\gamma m^2 r^2}\frac{d^2 u}{d\phi^2} \quad -(23)$$

and

$$\gamma r \dot{\phi}^2 = -\frac{L^2}{\gamma m^2 r^3} \quad -(24)$$

From eqs. (17), (23) and (24):

$$\boxed{\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma m r^2}{L^2} F(r)}$$

(25)

) This is the relativistic Binet equation of orbits. For

$$F(r) = -\frac{mMG}{r^2} \quad -(26)$$

$$\boxed{\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{\gamma}{\alpha}} \quad -(27)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(28)$$

and

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(29)$$

Eq. (27) is the same as obtained in previous UFT papers, Q.E.D. thus proving overall self-consistency.

In general:

$$\dot{r} = -\frac{L}{m\gamma}\frac{d}{d\phi}\left(\frac{1}{r}\right) \quad -(30)$$

and

$$\dot{\phi} = \frac{L}{\gamma m r^2} \quad -(31)$$

so

$$v^2 = \frac{L^2}{m^2\gamma^2}\left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2 + \frac{L^2}{\gamma^2 m^2 r^2}$$

$$= \frac{L^2}{m^2\gamma^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right) \quad -(32)$$

So:

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2}$$

$$= 1 - \frac{L^2}{\gamma^2 m^2 c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right) \quad -(33)$$

So:

$$\frac{1}{\gamma^2}\left(1 + \frac{L^2}{m^2c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right)\right) = 1 \quad -(34)$$

i.e.

$$\gamma^2 = 1 + \frac{L^2}{m^2c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right) \quad -(35)$$

From eqs. (27) and (35):

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1 + \frac{L^2}{m^2c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right)\right)^{1/2} \quad -(36)$$

Eq. (36) is the precise equation of the orbit, and is derived in this note for the first time.

If the orbital velocity v is observable then the orbit is given by:

$$v^2 = \frac{L^2\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right)}{m^2\left(1 + \frac{L^2}{m^2c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right)\right)} \quad -(37)$$

Eq. (36) can be written as:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+x\right)^{1/2} \quad -(38)$$

where

$$x=\frac{L^2}{m^2c^2}\left(\frac{1}{r^2}+\left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right) \quad -(39)$$

If

$$x \ll 1 \quad -(40)$$

then:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+\frac{x}{2}+\ldots\right) \quad -(41)$$

so to an excellent approximation for planetary orbits:

$$\boxed{\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+\frac{L^2}{2m^2c^2}\left(\frac{1}{r^2}+\left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right)\right) \quad -(42)}$$

This is a second order non-linear differential equation which can be compared with the results in UFT401, where it was shown that retrograde or forward precession depends on the sign of the spin connection.