402(6): Precise Definition of the Relativistic Lagrangian and Summary of Newtonian Mechanics

In Cartesian coordinates the Lagrangian is defined by:

$$\mathcal{L} = -mc^2\left(1-\left(\frac{v_x^2+v_y^2+v_z^2}{c^2}\right)\right)^{1/2} - U \quad -(1)$$

where

$$v^2 = v_x^2 + v_y^2 + v_z^2 \quad -(2)$$

is the speed of one reference frame with respect to another. The relativistic momentum is defined by:

$$p_i = \gamma m v_i \quad -(3)$$

where

$$i = X, Y, Z \quad -(4)$$

It follows that:

$$p_x = \frac{\partial \mathcal{L}}{\partial v_x} \quad -(5)$$

and so on, Q.E.D.

Summary of Newtonian Dynamics

The Newtonian orbital equation is:

$$\underline{F} = m\ddot{\underline{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(6)$$

This was developed in Cartesian coordinates in UFT377. In plane polar coordinates:

$$\underline{r} = r\,\underline{e}_r \quad -(7)$$

$$\ddot{\underline{r}} = (\ddot{r}-r\dot{\phi}^2)\,\underline{e}_r + (r\ddot{\phi}+2\dot{r}\dot{\phi})\,\underline{e}_\phi \quad -(8)$$

From eqs. (1) to (8) the following two equations are obtained:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} \quad -(9)$$

and

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(10)$$

2) Eq. (9) is the Leibnitz equation of orbits, and eq. (10) is the conservation of angular momentum. Eq. (9) is obtained from the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad - (11)$$

with the lagrangian:

$$\mathcal{L} = \frac{1}{2} m \underline{v} \cdot \underline{v} - U$$

$$= \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{mMG}{r} \quad - (12)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) + \frac{mMG}{r}$$

Eq. (10) is obtained from

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad - (13)$$

The generalized momentum from eq. (11) is the linear momentum:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad - (14)$$

The generalized momentum from eq. (13) is the angular momentum:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad - (15)$$

It follows from eq. (13) that:

$$\frac{dL}{dt} = 0 \quad - (16)$$

i.e. angular momentum is conserved.

3) The angular momentum is:

$$L = mr^2\dot{\phi}. \quad -(17)$$

It follows that

$$\dot{\phi} = \frac{L}{mr^2} \quad -(18)$$

$$\ddot{\phi} = \frac{d\dot{\phi}}{dt} = \frac{d}{dt}\left(\frac{L}{mr^2}\right)$$

$$= \frac{d}{dr}\left(\frac{L}{mr^2}\right)\frac{dr}{dt}$$

$$= -\frac{2\dot{r}L}{mr^3} \quad -(19)$$

It follows from eqs. (18) and (19) that:

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(20)$$

which is eq. (10), Q.E.D.

The Newtonian Hamiltonian is:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{mMG}{r} \quad -(21)$$

and is conserved:

$$\frac{dH}{dt} = 0 \quad -(22)$$

So H and L are constants of motion.

Using eq. (17):

$$\frac{dH}{dt} = \dot{r}\frac{dH}{dr} = \frac{1}{2}m\dot{r}\frac{d}{dr}\left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) + \frac{\dot{r}mMG}{r^2}$$

$$= 0 \quad -(23)$$

4) Using

$$\frac{d}{dr}\left(\dot{r}^2\right) = \frac{d}{dt}\left(\dot{r}^2\right)\frac{1}{\dot{r}} = 2\frac{\dot{r}\ddot{r}}{\dot{r}} = 2\ddot{r} \quad -(24)$$

the Leibnitz equation (9) follows:

$$\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{mG}{r^2} \quad -(25)$$

Therefore:

$$\boxed{\frac{dH}{dt} = 0} \longrightarrow \boxed{\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{mG}{r^2}} \longleftarrow \boxed{\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}}$$

$$\boxed{\frac{dL}{dt} = 0} \longrightarrow \boxed{r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0} \longleftarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}$$

The Leibnitz equation (9) may be developed as the Binet equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(26)$$

In the orbital half right latitude is:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(27)$$

By inspection, the conic section:

$$r = \frac{\alpha}{1 + \epsilon\cos\phi} \quad -(28)$$

obeys the Binet equation. So the Newtonian dynamics give the conic section.