02(5): Self Consistent Format for the Relativistic Newton Equation.

Consider the relativistic velocity:

$$\underline{v} = \gamma \frac{d\underline{r}}{dt} \quad -(1)$$

Then the relativistic acceleration is:

$$\underline{a} = \frac{d}{dt}\left(\gamma \dot{\underline{r}}\right) \quad -(2)$$

$$= \gamma \ddot{\underline{r}} + \dot{\underline{r}} \frac{d\gamma}{dt}$$

So the relativistic force equation is:

$$\underline{F} = m\left(\gamma \ddot{\underline{r}} + \dot{\underline{r}} \frac{d\gamma}{dt}\right) = -\frac{mMG}{r^3}\underline{r} \quad -(3)$$

In plane polar coordinates:

$$\ddot{\underline{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi \quad -(4)$$

and

$$\dot{\underline{r}} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(5)$$

So eq. (3) becomes:

$$\gamma\left(\ddot{r} - r\dot{\phi}^2\right) + \frac{d\gamma}{dt}\dot{r} = -\frac{MG}{r^2} \quad -(6)$$

and

$$\gamma\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) + r\dot{\phi}\frac{d\gamma}{dt} = 0 \quad -(7)$$

Eq. (6) is the relativistic Leibniz equation.

) Eq. (7) is the conservation of relativistic angular momentum:

$$\frac{dL}{dt} = 0 \quad -(8)$$

Use

$$\frac{dL}{dt} = \frac{d}{dt}(\gamma m r^2 \dot{\phi}) \quad -(9)$$

$$= 0$$

i.e.

$$\gamma \frac{d}{dt}(r^2 \dot{\phi}) + r^2 \dot{\phi} \frac{d\gamma}{dt} = 0 \quad -(10)$$

$$= \gamma \left(r^2 \ddot{\phi} + \frac{dr^2}{dt} \dot{\phi} \right) + r^2 \dot{\phi} \frac{d\gamma}{dt}$$

Now use:

$$\frac{dr^2}{dt} = \frac{dr^2}{dr} \frac{dr}{dt} = 2 \dot{r} r \quad -(11)$$

so eq. (10) becomes:

$$\gamma (r^2 \ddot{\phi} + 2 r \dot{r} \dot{\phi}) + r^2 \dot{\phi} \frac{d\gamma}{dt} \quad -(12)$$

$$= \gamma (r \ddot{\phi} + 2 \dot{r} \dot{\phi}) + r \dot{\phi} \frac{d\gamma}{dt}$$

$$= 0$$

Q.E.D., i.e. eq. (8) leads to eq. (7) in a rigorously self consistent way. Relativistic angular momentum is conserved.