402(3): The Generalized Momentum

The generalized momentum is defined by:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad -(1)$$

(Marion and Thornton eq. (6.151) of the third edition).

where

$$\mathcal{L} = T - U \quad -(2)$$

The Lagrange equations of motion follow immediately from eq. (1) by differentiation with respect to t:

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad -(3)$$

For a central potential U, for example:

$$U = \frac{-mMG}{(X^2+Y^2)^{1/2}} \quad -(4)$$

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} = \frac{dp_i}{dt} \quad -(5)$$

so

$$\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad -(6)$$

Q.E.D.

In note 402(2), Eq. (2), Ω_i is written as:

$$\frac{d\underline{p}}{dt} = \frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \quad -(7)$$

Note that eq. (1) is also the origin of

the relativistic Lagrangian as described by Marion and Thornton in Section 14.10 of the third edition. It is written as:

$$p_i = \frac{\partial \mathcal{L}}{\partial v_i} \quad — (8)$$

The Lagrangian is chosen to give the relativistic momentum

$$p_i = \gamma m v_i \quad — (9)$$

so

$$\frac{\partial \mathcal{L}}{\partial v_i} = \gamma m v_i \quad — (10)$$

so

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U$$

$$= -mc^2\left(1 - \frac{v_i^2}{c^2}\right)^{1/2} - U \quad — (11)$$

So eq. (1) is the origin both of the Lagrange equations and of the relativistic Lagrangian. Clearly eq. (1) is not independent of eq. (7) and eq. (11), it is the origin of these equations