402(2): Derivation of the force Equation from the Lagrangian

Non-Relativistic

The lagrangian is:

$$\mathcal{L} = T - V = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{mMG}{r} \quad -(1)$$

Consider the formal Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = \frac{d\underline{p}}{dt} \quad -(2)$$

in which

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \quad -(3)$$

from the fundamentals of the Hamilton and Lagrange dynamics. It is clear that eq. (3) follows from the formal differentiation of $\mathcal{L}$ with respect to $\underline{\dot{r}}$, i.e.

$$\frac{d}{d\dot{r}} \underline{\dot{r}} \cdot \underline{\dot{r}} = 2 \underline{\dot{r}} \quad -(4)$$

Therefore eq. (2) is the Newton orbital equation:

$$\underline{F} = \frac{d\underline{p}}{dt} = -mMG \frac{\underline{r}}{r^3} \quad -(5)$$

Q.E.D.

The formal equation (2) is defined in vector algebra as:

$$\frac{\partial \mathcal{L}}{\partial X} \underline{i} + \frac{\partial \mathcal{L}}{\partial Y} \underline{j} = \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \underline{i} + \frac{\partial \mathcal{L}}{\partial \dot{Y}} \underline{j} \right) \quad -(6)$$

$$= m \left( \ddot{X} \underline{i} + \ddot{Y} \underline{j} \right)$$

This definition gives rigorously self consistent results

as follows:
$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = m\ddot{X} \quad -(7)$$
$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} = m\ddot{Y} \quad -(8)$$

The lagrangian (1) is:
$$\mathcal{L} = \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{mMG}{(X^2+Y^2)^{1/2}} \quad -(9)$$

From eqs. (7) to (9):
$$F_X = m\ddot{X} = -mMG\frac{X}{(X^2+Y^2)^{1/2}} \quad -(10)$$
$$F_Y = m\ddot{Y} = -mMG\frac{Y}{(X^2+Y^2)^{1/2}} \quad -(11)$$

So
$$\underline{F} = F_X\,\underline{i} + F_Y\,\underline{j} = -mMG\frac{(X\underline{i} + Y\underline{j})}{(X^2+Y^2)^{3/2}} \quad -(12)$$
$$= \frac{d\underline{p}}{dt} = -mMG\frac{\underline{r}}{r^3}$$

Q.E.D. Eqs. (2) and (6) give the same result.

Relativistic

The relativistic lagrangian is:
$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(13)$$
$$= -mc^2\left(1 - \frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|}$$

3) using the formal equation (2) then:

$$\underline{F} = m\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = m\frac{d}{dt}(\gamma \underline{\dot{r}})$$

$$= \gamma^3 m \underline{\ddot{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(14)$$

which is the relativistic force equation Q.E.D.

The lagrangian (13) is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^{1/2} + \frac{mMG}{(x^2+y^2)^{1/2}} \quad -(15)$$

The formal equation (2) becomes eq. (5), so:

$$F_x = m\frac{d}{dt}(\gamma\dot{x}) = -mMG\frac{x}{(x^2+y^2)^{3/2}} \quad -(16)$$

$$F_y = m\frac{d}{dt}(\gamma\dot{y}) = -mMG\frac{y}{(x^2+y^2)^{3/2}} \quad -(17)$$

i.e.

$$\gamma^3\ddot{x} = -MG\frac{x}{(x^2+y^2)^{3/2}} \quad -(18)$$

$$\gamma^3\ddot{y} = -MG\frac{y}{(x^2+y^2)^{3/2}} \quad -(19)$$

which are the equations used in UFT377 Q.E.D.

Eqs. (16) and (17) are eqs. (14) in component form, Q.E.D. Therefore the theory is self consistent.

In summary:

The classical force equation is:

$$\underline{F} = m\underline{\ddot{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(20)$$

The relativistic force equation is:

$$\underline{F} = \gamma^3 m\underline{\ddot{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(21)$$

in which:

$$\gamma = \left(1 - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2}\right)^{-1/2} \quad -(22)$$

Eq. (21) gives orbital precession, eq. (20) does not.

Plane Polar Analysis

Eqs. (20) to (22) are true in any coordinate system. For plane polars:

$$\underline{r} = r\,\underline{e}_r \quad -(23)$$

$$\underline{\dot{r}} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(24)$$

$$\underline{\ddot{r}} = (\ddot{r} - r\dot{\phi}^2)\,\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\underline{e}_\phi \quad -(25)$$

Therefore in plane polars, eq. (20) gives:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{mMG}{r^2} \quad -(26)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(27)$$

and eq. (21) gives:

$$\gamma^3(\ddot{r} - r\dot{\phi}^2) = -\frac{mMG}{r^2} \quad -(28)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(29)$$

in which:

5) $$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2} \quad -(30)$$

Numerical integration of eqs. (26) and (27), regarded as simultaneous equations, must produce the orbit

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(31)$$

i.e. a conic section. Once this is demonstrated, numerical integration of the simultaneous equations (28) and (29) produce a precessing orbit (UFT 377).

Lagrangian in Plane Polars

This is described in Marion and Thornton, chapter 7 of classical dynamics. In this case:

$$\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{mMG}{r} \quad -(32)$$

with

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(33)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(34)$$

Eq. (34) produces the conserved angular momentum:

$$L = mr^2\dot{\phi} \quad -(35)$$

$$\frac{dL}{dt} = 0 \quad -(36)$$

Eq. (33) gives eq. (26).

Eqs. (27) or (29) are missing, and must be given by transforming eq. (6) to plane polars.

) The conserved angular momentum is:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(37)$$

$$\frac{d\underline{L}}{dt} = \underline{0} \quad -(38)$$

and can be expressed in terms of Cartesian coordinates:

$$\underline{L} = m(x\dot{y} - y\dot{x})\underline{k} \quad -(39)$$

$$\frac{d\underline{L}}{dt} = 0 \quad -(40)$$

So:

$$\frac{d\underline{L}}{dt} = m(\dot{x}\dot{y} + x\ddot{y} - (\dot{y}\dot{x} + x\ddot{y}))\underline{k}$$

$$= \underline{0} \quad -(41)$$

Q.E.D.

The relativistic lagrangian in plane polars is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(42)$$

So the relativistic angular momentum is conserved via the lagrangian (42) used in eq. (34):

$$L = \gamma m r^2 \dot{\phi} \quad -(43)$$

$$\frac{dL}{dt} = 0 \quad -(44)$$

Q.E.D

Eqs. (33) and (42) give:

$$F = m\frac{d}{dt}(\gamma\dot{r}) = -\frac{mMG}{r^2} \quad -(45)$$

i.e.

$$F = \gamma^3 m\ddot{r} = -\frac{mMG}{r^2} \quad -(46)$$

Q.E.D. Eq. (46) gives orbital precession with conservation of relativistic angular momentum.