402(1): The Origin of Retrograde Precession

As shown in UFT377 the retrograde precession of objects in orbit is given by the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \frac{d\underline{p}}{dt} \quad -(1)$$

where $\underline{r}$ is the position vector and $\underline{p}$ the relativistic momentum. By definition:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \underline{\nabla}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial X}\underline{i} + \frac{\partial \mathcal{L}}{\partial Y}\underline{j} + \frac{\partial \mathcal{L}}{\partial Z}\underline{k} \quad -(2)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \frac{\partial \mathcal{L}}{\partial \dot{X}}\underline{i} + \frac{\partial \mathcal{L}}{\partial \dot{Y}}\underline{j} + \frac{\partial \mathcal{L}}{\partial \dot{Z}}\underline{k} \quad -(3)$$

For a two dimensional orbit:

$$\frac{\partial \mathcal{L}}{\partial X}\underline{i} + \frac{\partial \mathcal{L}}{\partial Y}\underline{j} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\underline{i} + \frac{\partial \mathcal{L}}{\partial \dot{Y}}\underline{j}\right)$$

$$= \frac{dp_X}{dt}\underline{i} + \frac{dp_Y}{dt}\underline{j} \quad -(4)$$

$$= \underline{F}$$

where $\underline{F}$ is the relativistic Newtonian force. The latter is defined by:

$$\underline{F} = \frac{d\underline{p}}{dt} \quad -(5)$$

where

$$\underline{p} = \gamma m \underline{v} \quad -(6)$$

is the relativistic momentum. Here $\underline{v}$ is the Newtonian at non relativistic velocity, and the Lorentz factor is

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(7)$$

The relativistic Lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(8)$$

$$= -mc^2\left(1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|}$$

From eqs. (1) and (8):

$$\underline{F} = \frac{d\underline{p}}{dt} = -mMG\frac{\underline{r}}{r^3} \quad -(9)$$

The formal procedure leading from eq. (1) to eq. (9) is equivalent to eq. (4), from which:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \quad -(10)$$

and

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(11)$$

i.e.

$$F_X = \frac{dp_X}{dt} = -mMG\frac{X}{(X^2+Y^2)^{3/2}} \quad -(12)$$

$$F_Y = \frac{dp_Y}{dt} = -mMG\frac{Y}{(X^2+Y^2)^{3/2}} \quad -(13)$$

It is seen that eqs (12) and (13) mean the same thing as eq. (9), Q.E.D.

3) Here we have used:

$$\dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{X}^2 + \dot{Y}^2 \quad -(14)$$

The proper Lagrange variable is formally $\underline{r}$, the vector.

As in Note 377(1), the magnitude of the relativistic Newtonian force is:

$$F = \frac{d}{dt}(\gamma m v) = m \frac{d}{dt}(\gamma v) = m\left(v \frac{d\gamma}{dt} + \gamma \frac{dv}{dt}\right)$$

$$= m \frac{dv}{dt}\left(v \frac{d\gamma}{dv} + \gamma\right) \quad -(15)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(16)$$

so

$$\frac{d\gamma}{dv} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} = \gamma^3 \frac{v}{c^2} \quad -(17)$$

so

$$F = m \frac{dv}{dt}\left(\gamma^3 \frac{v^2}{c^2} + \gamma\right)$$

$$= \gamma m \frac{dv}{dt}\left(1 + \left(1 - \frac{v^2}{c^2}\right)^{-1} \frac{v^2}{c^2}\right)$$

$$= m \gamma \frac{dv}{dt}\left(\frac{c^2\left(1 - \frac{v^2}{c^2}\right) + v^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}\right) \quad -(18)$$

$$F = m \gamma^3 \frac{dv}{dt}.$$

This equation means that:

$$|\underline{F}| = m\gamma^3 \left|\frac{d\underline{v}}{dt}\right| \quad -(19)$$

i.e.

$$\left(F_x^2 + F_y^2 + F_z^2\right)^{1/2} = m\gamma^3 \left(\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_z}{dt}\right)^2\right)^{1/2} \quad -(20)$$

Square both sides:

$$F_x^2 + F_y^2 + F_z^2 = m^2\gamma^3 \left(\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_z}{dt}\right)^2\right) \quad -(21)$$

i.e.

$$\left(F_x \underline{i} + F_y \underline{j} + F_z \underline{k}\right) \cdot \left(F_x \underline{i} + F_y \underline{j} + F_z \underline{k}\right)$$
$$= m^2\gamma^6 \left(\frac{dv_x}{dt}\underline{i} + \frac{dv_y}{dt}\underline{j} + \frac{dv_z}{dt}\underline{k}\right) \cdot \left(\frac{dv_x}{dt}\underline{i} + \frac{dv_y}{dt}\underline{j} + \frac{dv_z}{dt}\underline{k}\right) \quad -(22)$$

It follows that:

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$$
$$= \gamma^3 m \left(\frac{dv_x}{dt}\underline{i} + \frac{dv_y}{dt}\underline{j} + \frac{dv_z}{dt}\underline{k}\right)$$
$$= \gamma^3 m \dot{\underline{v}} = \gamma^3 m \frac{d\underline{v}}{dt} \quad -(23)$$

So eq. (9) becomes:

$$\gamma^3 m \dot{\underline{v}} = -mMG\frac{\underline{r}}{r^3} = \gamma m \ddot{\underline{r}} \quad -(24)$$

and eqs. (12) and (13) become:

5)
$$\gamma^3 \ddot{X} = -mG \frac{X}{(X^2+Y^2)^{3/2}} \quad -(25)$$

$$\gamma^3 \ddot{Y} = -mG \frac{Y}{(X^2+Y^2)^{3/2}} \quad -(26)$$

where
$$\gamma = \left(1 - \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^{-1/2} \quad -(27)$$

As shown in UFT377, numerical solution of eqs. (25) and (26) give retrograde precession, Q.E.D. Einsteinian general relativity (EGR) cannot explain retrograde precession, which is observed experimentally in S star systems.

Notes
1) In going from eq. (18) to eq. (23), the Lorentz factor is a scalar, and remains the same.
2) As in UFT377, forward precession is given by solving eqs. (10) and (11) to give

$$\ddot{X} = \frac{mG}{\gamma(X^2+Y^2)^{3/2}}\left(\frac{\dot{X}\dot{Y}Y + X\dot{X}^2}{c^2} - X\right) \quad -(28)$$

and
$$\ddot{Y} = \frac{mG}{\gamma(X^2+Y^2)^{3/2}}\left(\frac{\dot{Y}\dot{X}X + Y\dot{Y}^2}{c^2} - Y\right) \quad -(29)$$

Retrograde precession emerges from the same eqs. (10) and (11) with the use of:

$$F_x = \frac{dp_x}{dt} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \quad - (30)$$

and $$F_y = \frac{dp_y}{dt} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \quad - (31)$$

where $$p_x = \gamma m v_x \quad - (32)$$

$$p_y = \gamma m v_y \quad - (33)$$

are relativistic momenta components. These are deduced from the law of conservation of momentum in special relativity (Marion and Thornton Chapter 14, 3rd ed.).

Therefore retrograde precession takes account of conservation of momentum, and forward precession uses only the lagrangian.